

UNIFORMIZATION BY SQUARE DOMAINS

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Dedicated to David Minda on the occasion of his retirement

ABSTRACT. Let Ω be a finitely connected subdomain of the Riemann sphere $\widehat{\mathbb{C}}$ with $\infty \in \Omega$. We find an extremal problem for conformal maps on Ω with suitable normalization at ∞ whose unique solution is a map onto a square domain, that is, a domain in $\widehat{\mathbb{C}}$ whose complementary components are (possibly degenerate) squares with sides parallel to the real or the imaginary axis.

1. INTRODUCTION

In the following, Ω will always denote a domain (that is, an open and connected set) in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $\infty \in \Omega$. We consider the family \mathcal{F} of all conformal maps $f: \Omega \rightarrow f(\Omega) \subseteq \widehat{\mathbb{C}}$ with the normalization

$$(1) \quad f(z) = z + \frac{a_1}{z} + \dots$$

for z near ∞ . It is a classical and well-known fact [Go, Chapter 5] that the functional

$$(2) \quad f \in \mathcal{F} \mapsto \mathcal{L}(f) := \operatorname{Re}(a_1)$$

has a unique minimum that is achieved by a conformal map $f = f_0: \Omega \rightarrow D$ onto a parallel slit domain $D \subseteq \widehat{\mathbb{C}}$ with slits parallel to the imaginary axis in \mathbb{C} . Here by definition a *parallel slit domain* D is a domain in $\widehat{\mathbb{C}}$ with $\infty \in D$ whose complementary components are closed line segments parallel to a fixed direction or single points (viewed as degenerate line segments). If one considers the more general functional

$$(3) \quad f \in \mathcal{F} \mapsto \mathcal{L}_\alpha(f) := \operatorname{Re}(e^{i\alpha} a_1)$$

for fixed $\alpha \in \mathbb{R}$, then it has a unique minimum given by a conformal map of Ω onto a parallel slit domain with slits parallel to the line $\ell_\alpha = \{ie^{i\alpha/2}t : t \in \mathbb{R}\}$.

Note that finding the maximum of \mathcal{L}_α is equivalent to finding the minimum of $\mathcal{L}_{\alpha+\pi}$. In particular, the functional $\mathcal{L} = \mathcal{L}_0$ has a unique maximum given by a conformal map onto a parallel slit domain with slits parallel to the real axis.

These facts are behind a conceptually simple proof of the statement that every domain Ω in \mathbb{C} can be uniformized by a parallel slit domain D (meaning that there exists a conformal map $f: \Omega \rightarrow D$): one first shows by a compactness argument that a minimum $f = f_0$ of the functional in (2) exists and then that f_0 maps onto a slit domain by a variational argument.

In contrast to this situation, it is an open problem whether each domain Ω can be uniformized by a *circle domain*, i.e., a domain in $\widehat{\mathbb{C}}$ whose complementary

Date: May 9, 2016.

The author was partially supported by NSF grant DMS-1162471.

components are single points or Euclidean disks. This is known as Koebe's "Kreismannormierungsproblem".

On the positive side, if Ω is finitely connected or, more generally, has countably many complementary components, then Ω can be uniformized by a circle domain (see [Go, Chapter 5] for the classical case of finitely connected domains and [HS] for domains with countably many complementary components). The proof of this fact is much more involved than the simple argument leading to uniformization by parallel slit domains as outlined above. No simple extremal problem is known that would solve the "Kreismannormierungsproblem", even in the finitely connected case.

The main point of the present note is the observation that there is such an extremal problem for the uniformization of finitely connected domains onto square domains. By definition a *square domain* D is a domain in $\widehat{\mathbb{C}}$ with $\infty \in D$ whose complementary components are single points or closed squares with sides parallel to the real or the imaginary axis.

To formulate this extremal problem, we fix some notation. If $f \in \mathcal{F}$ is a conformal map defined on a finitely connected Ω as above, we denote by $D = f(\Omega)$ its image domain and by K_j , $j = 1, \dots, n$, the complementary components of D in $\widehat{\mathbb{C}}$. We allow $n = 0$ here. Then $D = \Omega = \widehat{\mathbb{C}}$ and the class \mathcal{F} just consists of the identity map. We could ignore components of Ω and D that are points, because an isolated point in the complement of Ω is a removable singularity. For simplicity, we allow such components.

Each set K_j is a compact subset of \mathbb{C} . We denote by A_j the Euclidean area (that is, the Lebesgue measure) of K_j and by V_j its *vertical variation* defined as

$$V_j = \max_{w \in K_j} \operatorname{Im}(w) - \min_{w \in K_j} \operatorname{Im}(w).$$

Of course, the quantities A_j and V_j depend on f , but for simplicity we suppress this dependence in our notation.

We can now state our main theorem.

Theorem 1.1. *Let Ω be a finitely connected domain in $\widehat{\mathbb{C}}$ with $\infty \in \Omega$. Among all conformal maps $f: \Omega \rightarrow D = f(\Omega) \subseteq \widehat{\mathbb{C}}$ with the normalization*

$$(4) \quad f(z) = z + \frac{a_1}{z} + \dots$$

the functional

$$(5) \quad f \in \mathcal{F} \mapsto \mathcal{S}(f) := 2\pi \operatorname{Re}(a_1) + \sum_{j=1}^n (V_j^2 - A_j)$$

has a unique minimizer $f = f_0 \in \mathcal{F}$. The map f_0 is the unique conformal $f_0: \Omega \rightarrow D$ map with the normalization (4) onto a square domain D .

2. PROOF OF THEOREM 1.1

Let Ω be as in Theorem 1.1. It is a known fact that there exists a conformal map g of Ω onto a square domain $\tilde{\Omega}$ with the normalization

$$g(z) = z + \frac{b_1}{z} + \dots$$

near ∞ (this follows from the Brandt-Harrington Uniformization Theorem; for the formulation of this theorem and its proof see [S96]). We denote by $\tilde{\mathcal{F}}$ the class of

conformal maps f on $\tilde{\Omega}$ normalized as in (4), and by $\tilde{\mathcal{S}}$ the functional on $\tilde{\mathcal{F}}$ defined as in (5). The map $f \in \mathcal{F} \mapsto \tilde{f} := f \circ g^{-1}$ is a bijection between \mathcal{F} and $\tilde{\mathcal{F}}$. Moreover, if

$$f(z) = z + \frac{a_1}{z} + \dots \text{ and } \tilde{f}(z) = z + \frac{\tilde{a}_1}{z} + \dots$$

near ∞ , then $\tilde{a}_1 = a_1 - b_1$. Since the maps f and \tilde{f} have the same image domain $D = f(\Omega) = \tilde{f}(\tilde{\Omega})$, we have

$$\tilde{\mathcal{S}}(\tilde{f}) = \mathcal{S}(f) - 2\pi \operatorname{Re}(b_1).$$

So under the bijection $f \leftrightarrow \tilde{f}$, the functionals \mathcal{S} and $\tilde{\mathcal{S}}$ correspond to each other and just differ by the fixed additive constant $-2\pi \operatorname{Re}(b_1)$. In this way, the proof of Theorem 1.1 is reduced to the case where Ω is a square domain to begin with.

The theorem will now follow from the following statement.

Proposition 2.1. *Let Ω be a finitely connected square domain in $\hat{\mathbb{C}}$ with $\infty \in \Omega$. For all conformal maps $f: \Omega \rightarrow D := f(\Omega) \subseteq \hat{\mathbb{C}}$ with the normalization (4) we have*

$$(6) \quad \mathcal{S}(f) = 2\pi \operatorname{Re}(a_1) + \sum_{j=1}^n (V_j^2 - A_j) \geq 0$$

with equality if and only if f is the identity on Ω .

Moreover, the identity on Ω is the only conformal map f of Ω with the normalization (4) onto a square domain.

Proof. Before we show the main inequality (6), let us convince ourselves how the last uniqueness statement can be derived from the uniqueness statement in (6).

Indeed, suppose $f: \Omega \rightarrow \Omega'$ is a conformal map onto another square domain Ω' . The map f^{-1} on Ω' is also normalized as in (4), where the coefficients a'_1 of f^{-1} and a_1 of f are related by the equation $a'_1 = -a_1$.

Since the roles of Ω and Ω' are symmetric, we may assume that $\operatorname{Re}(a_1) \leq 0$; for otherwise, we consider f^{-1} instead of f .

Since $\Omega' = f(\Omega)$ is a square domain, we have $V_j^2 = A_j$ for $j = 1, \dots, n$. Hence $\mathcal{S}(f) = \operatorname{Re}(a_1) \leq 0$, and so $\mathcal{S}(f) = 0$ by (6). Now from the uniqueness statement in (6) we deduce that f is the identity on Ω as desired.

To prove (6), let f be as in the statement. We consider the rectangle $\mathcal{R} = [-l, l] \times [-r, r] \subset \mathbb{R}^2 \cong \mathbb{C}$ for large $r > 0$. Here we chose $l = r^{2/3}$ so that

$$(7) \quad l/r \rightarrow 0 \text{ and } r/l^2 \rightarrow 0$$

as $r \rightarrow \infty$.

In the following, we assume that r is so large that $\hat{\mathbb{C}} \setminus \Omega$ is contained in the interior of \mathcal{R} . Then $\partial\mathcal{R} \subseteq \Omega$ and $J = f(\partial\mathcal{R})$ is a Jordan curve in \mathbb{C} . We want to estimate the area $A = A(r)$ of the region enclosed by the positively oriented Jordan curve $J = f(\partial\mathcal{R})$ up to a term $o(1)$ as $r \rightarrow \infty$.

By the second relation in (7) we have

$$\begin{aligned}
A &= \frac{1}{2i} \int_J \bar{w} dw = \frac{1}{2i} \int_{\partial\mathcal{R}} \overline{f(z)} f'(z) dz \\
&= \frac{1}{2i} \int_{\partial\mathcal{R}} \overline{\left(z + \frac{a_1}{z} + \dots\right)} \left(1 - \frac{a_1}{z^2} + \dots\right) dz \\
&= \frac{1}{2i} \int_{\partial\mathcal{R}} \left(\bar{z} + \frac{\bar{a}_1}{\bar{z}} - \frac{a_1 \bar{z}}{z^2} + O\left(\frac{1}{|z|^2}\right)\right) dz \\
&= 4rl + \int_{\partial\mathcal{R}} \operatorname{Im}\left(\frac{\bar{a}_1 z}{\bar{z}}\right) \frac{dz}{z} + o(1).
\end{aligned}$$

It remains to estimate the last integral in this equation. Note that for this we may ignore its imaginary part, because the expressions A and $4rl$ are real.

We set $a_1 = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\gamma(t) = l + it$ for $t \in [-r, r]$. Then by the first relation in (7) we have

$$\begin{aligned}
\int_{\partial\mathcal{R}} \operatorname{Im}\left(\frac{\bar{a}_1 z}{\bar{z}}\right) \frac{dz}{z} &= 2 \int_{\gamma} \operatorname{Im}\left(\frac{\bar{a}_1 z}{\bar{z}}\right) \frac{dz}{z} + o(1) \\
&= 2 \int_{-r}^r \operatorname{Im}\left(\frac{\bar{a}_1 \gamma(t)}{\gamma(t)}\right) \frac{i(l - it)}{l^2 + t^2} dt + o(1) \\
&= 2 \int_{-r}^r \frac{2\alpha t - \beta(l^2 - t^2)}{l^2 + t^2} \cdot \frac{t}{l^2 + t^2} dt + o(1) \\
&= 4\alpha \int_{-r}^r \frac{lt^2}{(l^2 + t^2)^2} dt + o(1) \\
&= 4\alpha \int_{-r/l}^{r/l} \frac{u^2}{(1 + u^2)^2} du + o(1) \\
&= 4\alpha \int_{-\infty}^{+\infty} \frac{u^2}{(1 + u^2)^2} du + o(1) = 2\pi \operatorname{Re}(a_1) + o(1).
\end{aligned}$$

Hence

$$(8) \quad A = 4lr + 2\pi \operatorname{Re}(a_1) + o(1)$$

as $r \rightarrow \infty$.

For $j = 1, \dots, n$ we denote by S_j the complementary components of Ω and, as before, by K_j the complementary components of $D = f(\Omega)$. We may assume that the labels are chosen such that these components correspond to each other in the sense that if $z \in \Omega \rightarrow \partial S_j$, then $f(z) \rightarrow \partial K_j$ for $j = 1, \dots, n$.

The components S_j are squares (or possibly points as degenerate squares) with sides parallel to the real or the imaginary axis. Let $l_j \in [0, \infty)$ be the side length of S_j . We now define a Borel function $\rho: \mathbb{C} \rightarrow [0, \infty]$ as follows:

$$(9) \quad \rho(z) = \begin{cases} |f'(z)| & \text{for } z \in \Omega, \\ V_j/l_j & \text{for } z \in S_j \text{ and } l_j > 0, \\ \infty & \text{for } z \in S_j \text{ and } l_j = 0. \end{cases}$$

The rectangle \mathcal{R} contains all the squares S_j in its interior. So by (8) we have

$$\begin{aligned}
(10) \quad \int_{\mathcal{R}} \rho^2 &= \int_{\mathcal{R} \cap \Omega} |f'|^2 + \sum_{j=1}^n V_j^2 \\
&= \text{Area}(f(\mathcal{R} \cap \Omega)) + \sum_{j=1}^n V_j^2 \\
&= \text{Area}(f(\mathcal{R} \cap \Omega)) + \sum_{j=1}^n A_j + \sum_{j=1}^n (V_j^2 - A_j) \\
&= A + \sum_{j=1}^n (V_j^2 - A_j) \\
&= 4lr + \mathcal{S}(f) + o(1)
\end{aligned}$$

as $r \rightarrow \infty$. Here and in similar integrals below, integration is with respect to Lebesgue measure, and $\text{Area}(M)$ denotes the Lebesgue measure of a Borel set $M \subseteq \mathbb{C}$.

For $x \in [-l, l]$ let ℓ_x be the line segment $\ell_x = \{x + it : t \in [-r, r]\}$. Then we have

$$\begin{aligned}
(11) \quad \int_{\ell_x} \rho(z) |dz| &= \int_{\ell_x \cap \Omega} |f'(z)| |dz| + \sum_{S_j \cap \ell_x \neq \emptyset} V_j \\
&\geq \text{Im}(f(x + ir) - f(x - ir)) = 2r + O\left(\frac{1}{r}\right).
\end{aligned}$$

Here we used the fact that the union of the set $f(\ell_x \cap \Omega)$ together with all sets K_j with $S_j \cap \ell_x \neq \emptyset$ forms a connected set joining the points $f(x + ir)$ and $f(x - ir)$.

It follows that

$$\left(\int_{\ell_x} \rho(z) |dz| \right)^2 \geq 4r^2 + O(1),$$

and using (7) and (10) we conclude

$$\begin{aligned}
(12) \quad 4lr + o(1) &\leq \frac{1}{2r} \int_{-l}^l \left(\int_{\ell_x} \rho(z) |dz| \right)^2 dx \\
&\leq \int_{\mathcal{R}} \rho^2 = 4lr + \mathcal{S}(f) + o(1).
\end{aligned}$$

This implies $\mathcal{S}(f) \geq o(1)$ as $r \rightarrow \infty$, and so $\mathcal{S}(f) \geq 0$ as desired.

Suppose we have the equality $\mathcal{S}(f) = 0$. Then

$$(13) \quad \int_{\mathcal{R}} \rho^2 = 4lr + o(1)$$

as $r \rightarrow \infty$. If we define $\tilde{\rho} = (1 + \rho)$, then by (11) we have

$$\int_{\ell_x} \tilde{\rho}(z) |dz| \geq 4r + O\left(\frac{1}{r}\right),$$

and so

$$\begin{aligned} 16lr + o(1) &\leq \frac{1}{2r} \int_{-l}^l \left(\int_{l_x} \tilde{\rho}(z) |dz| \right)^2 dx \\ &\leq \int_{\mathcal{R}} \tilde{\rho}^2. \end{aligned}$$

Hence by (13),

$$\int_{\mathcal{R}} (1 - \rho)^2 = \int_{\mathcal{R}} (2 + 2\rho^2 - \tilde{\rho}^2) \leq 8lr + 8lr - 16lr + o(1) = o(1).$$

Letting $r \rightarrow \infty$, we conclude that $\int_{\mathbb{C}} (1 - \rho)^2 = 0$ and so $\rho = 1$ almost everywhere on \mathbb{C} . In particular, $|f'(z)| = 1$ for all $z \in \Omega$, and so f' is constant. With the given normalization this implies $f' \equiv 1$ and so f is the identity on Ω . \square

The proof of Theorem 1.1 is complete.

3. REMARKS

1. It is not clear which conformal map should maximize the functional in (5).
2. It would be very interesting to see whether the functional (5) can be used to give an independent existence proof for a conformal map of a finitely connected domain Ω onto a square domain D without resorting to the Brandt-Harrington Uniformization Theorem. By a normal family argument one can show the existence of a minimizer, but there seems to be no simple variational argument in the class of *conformal maps* that shows that the minimizer is a conformal map onto a square domain. One can formulate a more general variational problem for real-valued functions v (corresponding to the imaginary parts of the conformal maps) whose solution should give the existence of a suitable conformal map. This is in the same spirit as classical potential-theoretic methods for the solution of uniformization problems (see [Cou]).
3. Behind the proof of Proposition 2.1 is essentially an asymptotic estimate for the conformal modulus (or extremal length) of a path family (see [Ahl, Chapter 4]), namely the family of line segments parallel to the imaginary axis and joining the top to the bottom side of the rectangle \mathcal{R} . The use of the test function in (9) was inspired by Schramm's notion of transboundary extremal length as introduced in [S95].
4. Using asymptotic estimates for modulus in conformal mapping theory is a fairly standard idea. For example, the notion of reduced extremal length as discussed in [Ahl, Section 4.14] is closely related to this. The idea of using a rectangle with side lengths satisfying the conditions in (7) seems to be new.
5. Very similar arguments as in this note can be used to show the following (essentially known) fact.

Proposition 3.1. *Let Ω be a finitely connected parallel slit domain with slits parallel to the imaginary axis. Then $\operatorname{Re}(a_1) \geq 0$ for all conformal map $f: \Omega \rightarrow f(\Omega) \subseteq \widehat{\mathbb{C}}$ normalized as in (1) with equality if and only if f is the identity on Ω .*

Proof. We use the notation as in the proof of Proposition 2.1, but set $\rho = |f'|$ on Ω and $\rho = 0$ elsewhere. Then

$$\int_{\mathcal{R}} \rho^2 = \operatorname{Area}(f(\mathcal{R} \cap \Omega)) \leq A = 4lr + 2\pi \operatorname{Re}(a_1) + o(1)$$

as $r \rightarrow \infty$. We have $\ell_x \subseteq \Omega$ for every $x \in [-r, r]$ with at most finitely many exceptions. For these values x we have an estimate as in (11) and obtain a lower bound for $\int_{\mathcal{R}} \rho^2$ as in (12). The desired inequality $\operatorname{Re}(a_1) \geq 0$ follows. In case of equality we have

$$\int_{\mathcal{R}} \rho^2 \leq 4lr + o(1).$$

As in the last part of the proof of Proposition 2.1, this leads to $\rho = 1$ almost everywhere on \mathbb{C} , and so f must be the identity on Ω . \square

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